**A MULTIVARIATE GAMMA DISTRIBUTION AND ITS CHARACTERIZATIONS**

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SYNOPTIC ABSTRACT

In this paper, we proffer a new multivariate gamma distribution with potential applications in survival and reliability modeling. The multivariate distribution is not necessarily restricted to those with gamma marginal distributions. We provide and characterize a generalized location scale family of multivariate gamma distributions. This family possesses three-parameter gamma marginals (in most cases) and it contains absolutely continuous classes, as well as, the Marshall Olkin type of distributions with a positive probability mass on a set of measure zero. Maximum likelihood estimators are developed in the bivariate case and applied to real data.

**Key Words and Phrases:** survival, bivariate exponential, Weibull, maximum likelihood.

## 1. INTRODUCTION

When more that one time-to-event can be observed for each individual under study, such as time to multiple organ failure in diabetic patients, it is reasonable to assume that event times are dependent. However, it is common practice to assume "working independence" by analyzing each event separately and treating them as independent events (see Lawless (2003)). We will demonstrate that ignoring dependence can bring great costs in terms of bias, MSE and other objective estimation and efficiency criteria. For example, in terms of maximum likelihood estimation (MLE), the working independence approach systemically maximizes the wrong joint likelihood function. That is, the MLE of the parameters based on the marginal likelihoods are not the same as those derived from the joint likelihood.

Moreover, separate analysis on the correlated variables often leads to confusing interpretations. Also, in taking such an approach, one must make multiple testing adjustments that could lead to a decreased sensitivity to real differences in survival/reliability between groups. For example, in clinical trials, often more than one primary endpoint is to be examined, such as time-to-disappearance and time-to-recurrence of the tumor. In these situations, the FDA typically mandates Bonferroni-type adjustments for multiple testing.

In this paper, we define and characterize a multivariate gamma distribution, where the marginal's, under certain assumptions, are three-parameter (location/scale/shape) gamma random variables. The variables are linearly related through a collection of latent random variables.

## 2. GENERALIZED MULTIVARIATE GAMMA

In this paper, we say a random variable $X$ has a three-parameter gamma distribution, denoted as $X \sim Ga(\lambda, \alpha, \mu)$, if it possesses the following pdf

$$f_X(t;\mu,\lambda,\alpha) = \frac{\lambda^\alpha}{\Gamma(\alpha)}(t-\mu)^{\alpha-1}e^{-\lambda(t-\mu)}I_{[\mu,\infty)}(t) \tag{1}$$

where $\mu \in \mathbb{R}$, $\lambda > 0$, and $\alpha > 0$ represent the location, scale and shape parameters, respectively.

**Definition 1** Let $X_0, X_1, ..., X_p$ be marginally distributed as $Ga(\alpha_i, \lambda_i, \mu_i)$, $i = 0, 1, \cdots, p$, as in (1). Let $Z_1, \ldots, Z_p$ be $p$ mutually independent latent random variables satisfying the following linear condition

$$X_i = a_i X_0 + Z_i, \tag{2}$$

where $a_i > 0$ is some constant and $Z_i$ is independent with respect to $X_0$, $i = 1, \cdots, p$. We define the joint distribution of the random vector $\mathbf{X} = (X_1, X_2, ..., X_p)'$ as the multivariate gamma distribution, denoted as $\mathbf{X} = MVGa(\mu_i, \lambda_i, \alpha_i, i = 1, 2, .., p)$.

Note that the dependence structure between the observable variables $X_1, \ldots, X_p$ are defined linearly through a common latent variable $X_0$ and respective independent latent variables $Z_1, \ldots, Z_p$. Also, note that while we require that $X_0$ be a gamma random variable, no such restriction is placed on the latent variables $Z_1, \ldots, Z_p$.

Based on the above definition and well-known gamma random variable properties, it is straightforward to show that

$$E(X_i^m) = \sum_{k=0}^{m} \binom{m}{k} \lambda_i^{\alpha+k} \mu_i^{m-k} (\alpha_i)_k, (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1). \tag{3}$$

From the linear structure (2) we can express equation (3) as

$$E(X_i^m) = E(a_i X_0 + Z_i)^m = \sum_{r=0}^{m} \binom{m}{r} a_i^r E X_0^r E Z_i^{m-r}$$

$$= \sum_{r=0}^{m} \binom{m}{r} E Z_i^{m-r} a_i^r \sum_{k=0}^{r} \binom{r}{k} \lambda_i^{\alpha+k} \mu_i^{r-k} (\alpha_i)_k. \tag{4}$$

Similarly, we have the higher cross-product moments

$$E(X_i^m X_j^l) = E(a_i X_0 + Z_i)^m (a_j X_0 + Z_j)^l = E \sum_{r=0}^{m} \binom{m}{r} a_i^r X_0^r Z_i^{m-r} \sum_{s=0}^{l} \binom{l}{s} a_j^s X_0^s Z_j^{l-s} =$$

$$= \sum_{r=0}^{m} \sum_{s=0}^{l} \binom{m}{r} \binom{l}{s} a_i^r a_j^s E X_0^{r+s} E Z_i^{m-r} E Z_j^{l-s} \tag{5}$$

Equations (3), (4) and (5) directly give us

- $E(X_i) = \frac{\alpha_i}{\lambda_i} + \mu_i$,
- $Var(X_i) = \frac{\alpha_i}{\lambda_i^2}$,
- $Cov(X_i, X_j) = a_i a_j Var(X_0) = a_i a_j \frac{\alpha_0}{\lambda_0^2}$,

- $\rho(X_i, X_j) = a_i a_j \frac{\lambda_i \lambda_j}{\lambda_0^2} \frac{\alpha_0}{\sqrt{\alpha_i \alpha_j}}$.

More concisely, the mean vector is given as

$$E(\mathbf{X}) = \left[ \frac{\alpha_1}{\lambda_1} + \mu_1, \frac{\alpha_2}{\lambda_2} + \mu_2, \cdots, \frac{\alpha_p}{\lambda_p} + \mu_p \right]',$$

the variance/covariance matrix is given as

$$\Sigma = \begin{pmatrix} \frac{\alpha_1}{\lambda_1^2} & a_1 a_2 \frac{\alpha_0}{\lambda_0^2} & \cdots & a_1 a_p \frac{\alpha_0}{\lambda_0^2} \\ a_2 a_1 \frac{\alpha_0}{\lambda_0^2} & \frac{\alpha_2}{\lambda_2^2} & \cdots & a_2 a_p \frac{\alpha_0}{\lambda_0^2} \\ \vdots & \vdots & \vdots & \vdots \\ a_p a_1 \frac{\alpha_0}{\lambda_0^2} & \cdots & a_p a_{p-1} \frac{\alpha_0}{\lambda_0^2} & \frac{\alpha_p}{\lambda_p^2} \end{pmatrix},$$

and the correlation matrix is given as

$$\rho = \begin{pmatrix} 1 & \left(\frac{\alpha_0}{\lambda_0^2}\right) \frac{a_1 a_2 \lambda_1 \lambda_2}{\sqrt{\alpha_1 \alpha_2}} & \cdots & \left(\frac{\alpha_0}{\lambda_0^2}\right) \frac{a_1 a_p \lambda_1 \lambda_p}{\sqrt{\alpha_1 \alpha_p}} \\ \left(\frac{\alpha_0}{\lambda_0^2}\right) \frac{a_2 a_1 \lambda_2 \lambda_1}{\sqrt{\alpha_1 \alpha_2}} & 1 & \cdots & \left(\frac{\alpha_0}{\lambda_0^2}\right) \frac{a_2 a_p \lambda_2 \lambda_p}{\sqrt{\alpha_2 \alpha_p}} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\alpha_0}{\lambda_0^2}\right) \frac{a_p a_1 \lambda_p \lambda_1}{\sqrt{\alpha_p \alpha_1}} & \cdots & \left(\frac{\alpha_0}{\lambda_0^2}\right) \frac{a_p a_{p-1} \lambda_p \lambda_{p-1}}{\sqrt{\alpha_p \alpha_{p-1}}} & 1 \end{pmatrix}.$$

Although, the distributions of the latent variables, $Z_1, \ldots, Z_p$ are not yet specified, we can derive the following properties:

- $E(Z_i) = E(X_i) - a_i E(X_0) = (\frac{\alpha_i}{\lambda_i} - a_i \frac{\alpha_0}{\lambda_0}) + (\mu_i - a_i \mu_0)$,
- $Var(Z_i) = Var(X_i) - a_i^2 Var(X_0) = \frac{\alpha_i}{\lambda_i^2} - a_i^2 \frac{\alpha_0}{\lambda_0^2} > 0 \Longleftrightarrow 0 < a_i < \frac{\lambda_0}{\lambda_i} \sqrt{\frac{\alpha_i}{\alpha_0}}$,
- $Cov(Z_i, Z_j) = 0$.

If we let $a_i = \lambda_0 / \lambda_i, i = 1, \ldots, p$, then the above multivariate model reduces to the one first proposed by Mathai and Moschopoulos (1991, 1992) and later featured in Kotz, Balakrishnan and Johnson (2000). Interestingly, in this case, letting $Z_i \sim Ga(\lambda_i, \alpha_i - \alpha_0, \mu_i - \lambda_0 \mu_0 / \lambda_i)$, $i = 1, \cdots, p$, satisfies the conditions given in Definition 1 and the linear structure in (2) holds. In fact, Carpenter and Diawara (2006) show that the $a_i = \lambda_0 / \lambda_i, i = 1, \ldots, p$ is a necessary and sufficient condition for $Z_i$ to be gamma distributed. For any other values of $a_1, \ldots, a_p$ the latent variables cannot be gamma distributed. This fact is easily demonstrated deriving the Laplace transform for $Z_i$.

First, the Laplace transform for any $Ga(\lambda, \alpha, \mu)$ is given by

$$L(s) = e^{-\mu s}(1 + s/\lambda)^{-\alpha}. \tag{6}$$

However, since $X_i = aX_0 + Z_i$,

$$L_{X_i}(s) = Ee^{-sX_i} = Ee^{-s(aX_0+Z_i)} = L_{X_0}(as)L_{Z_i}(s) \implies L_{Z_i}(s) = L_{X_i}(s)/L_{X_0}(as). \tag{7}$$

Therefore, from (6) and (7) we have that

$$Ł_Z(s) = e^{-s(\mu_i - a\mu_0)}\frac{(1 + as/\lambda_0)^{\alpha_0}}{(1 + s/\lambda_i)^{\alpha_i}}. \tag{8}$$

If we let $a = \lambda_0/\lambda_i$, the result follows directly from (8) since in that case

$$Ł_Z(s) = e^{-s(\mu_i - \lambda_0\mu_0/\lambda_1)}(1 + s/\lambda_i)^{\alpha_i - \alpha_0}.$$

Carpenter, Diawara and Han (2006), Iyer, et al (2002) and Iyer and Manjunath (2004) show that for $X_i$ to have a marginal exponential distribution (a special case of the gamma), $Z_i$ must be a product of a Bernoulli and an exponential random variable, i.e., a mixture of a point mass at $Z = 0$ and an exponential. For illustrative purposes, we take up this special case in the next section.

## 3. A DIRECT LINEAR ASSOCIATION

In this section, we give an important special case of the model proposed in Section 2, the bivariate exponential case ($\alpha = 1, \mu = 0$). Carpenter, Diawara and Han (2006) develop a bivariate exponential model based on a method proposed by Iyer, et al (2002) and Iyer and Manjunath (2004). More specifically, we let $X_1$ and $X_2$ be fixed marginally as exponential random variables with hazard rates $\lambda_1$ and $\lambda_2$, respectively:

$$f_{X_i}(x) = \lambda_i e^{-\lambda_i x} I(x > 0), i = 1, 2. \tag{9}$$

Then by introducing a latent non-negative variable, $Z$, statistically independent of $X_1$, a linear relationship is formed between $X_1$ and $X_2$ by setting

$$X_2 = aX_1 + Z, \tag{10}$$

for $a > 0$. From (8), we see that the Laplace transform for $Z$ is given as

$$Ł_Z(s) = \frac{(1 + as/\lambda_1)}{(1 + s/\lambda_2)} = p + (1 - p)L_{X_2}(s), \tag{11}$$

where $0 < p = a\lambda_2/\lambda_1 < 1$. From (11), we see that the linear structure given in (10) forces $Z$ to be the product of a Bernoulli random variable and an exponential random variable. This product gives us a mixture of a point mass (with $P(Z = 0) = p$) at zero and the following relationship,

$$X_2 = aX_1 + Z = \begin{cases} aX_1 & \text{if } Z = 0 \\ aX_1 + Z & \text{if } Z > 0 \end{cases},$$

where $P(Z = 0) = p$, $P(Z > 0) = 1 - p$, $p = a\lambda_2/\lambda_1$, and $Z$ is independent of $X_1$. Also, the covariance/correlation between $X_1$ and $X_2$ easily found as

$$Cov(X_1, X_2) = a/\lambda_1^2 \qquad and \qquad Corr(X_1, X_2) = a\lambda_2/\lambda_1.$$

In Theorem 1, below, we give the maximum likelihood estimators (MLE) of $\lambda_1$ and $\lambda_2$ based on the joint likelihood expression. We refer to these estimators as $\hat{\lambda}_1$ and $\hat{\lambda}_2$ and these will be compared to the marginal MLE's, denoted as $\hat{\lambda}_1^*$ and $\hat{\lambda}_2^*$. We define the marginal MLE's as those estimators that maximize the univariate marginal likelihood functions separately for $\lambda_1$ and $\lambda_2$. Lawless (2003) refers to the analysis of the marginal MLE's as assuming "working independence". As we will show later, assuming working independence comes at a cost in terms of mean squared error (MSE).

**Theorem 1:** For a given random sample of size $n$, $(x_{1i}, x_{2i}), i = 1, \ldots, n,$, the joint maximum likelihood estimator of $(\lambda_1, \lambda_2)$ is $(\hat{\lambda}_1, \hat{\lambda}_2)$, where

$$\hat{\lambda}_1 = \frac{a}{\bar{x}_2} + \frac{(n - k)}{n\bar{x}_1} \text{ and } \hat{\lambda}_2 = \frac{1}{\bar{x}_2} \tag{12}$$

$k = \sum_{i=1}^{n} I(x_2 - ax_1 = 0)$, $\bar{x}_1 = \sum x_1/n$ and $\bar{x}_2 = \sum x_2/n$. Also, $\hat{\lambda}' = (\hat{\lambda}_1, \hat{\lambda}_2)'$ is approximately bivariate normal with mean vector $\lambda$ and variance/covariance matrix $\Sigma$, where

$$\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \text{ and } \Sigma = \frac{1}{n} \begin{pmatrix} \lambda_1(\lambda_1 - a\lambda_2) + a^2\lambda_2^2 & a\lambda_2^2 \\ a\lambda_2^2 & \lambda_2^2 \end{pmatrix}$$

and $\mathrm{Corr}(\hat{\lambda}_1, \hat{\lambda}_2) = ((1-p)/p^2 + 1)^{-1/2}$.

The alternatives to the joint MLE's given in (12) can by found by maximizing the marginal likelihood expressions separately for $\lambda_1$ and $\lambda_2$. These marginal MLE's are well-known (see page 54 of Lawless (2003)) and are given as

$$\hat{\lambda}_1^* = \frac{1}{\bar{x}_1} \qquad \text{and} \qquad \hat{\lambda}_2^* = \frac{1}{\bar{x}_2}. \tag{13}$$

We observe that the marginal and joint MLE's for $\lambda_2$ are identical, i.e., $\hat{\lambda}_2^* \equiv \hat{\lambda}_2$, but for $\lambda_1$ the MLE's are quite different for this model.

Table 1 summarizes the results of a simulation study where 10,000 simulated samples of size 25 pairs from a BVE$(1, 1, a)$ were generated.

**Table 1: Simulation Study for BVE($\lambda_1 = \lambda_2 = 1$, $n = 25$)**

| | Mean Squared Error (MSE) | | | | | |
|---|---|---|---|---|---|---|
| $\rho$ | $\hat{\lambda}_1^*$ | $\hat{\lambda}_1$ | %-imp | $\hat{\rho}_1^*$ | $\hat{\rho}_1$ | %-imp |
| 0.01 | 0.04585 | 0.04621 | 0.782 | 0.00001 | 0.00001 | 0.529 |
| 0.05 | 0.04522 | 0.04723 | 4.251 | 0.00021 | 0.00022 | 2.990 |
| 0.10 | 0.04512 | 0.05005 | 9.849 | 0.00077 | 0.00082 | 6.410 |
| 0.20 | 0.04118 | 0.04877 | 15.570 | 0.00252 | 0.00284 | 11.243 |
| 0.30 | 0.03930 | 0.04937 | 20.406 | 0.00456 | 0.00546 | 16.563 |
| 0.40 | 0.03627 | 0.04749 | 23.615 | 0.00634 | 0.00815 | 22.176 |
| 0.50 | 0.03757 | 0.04834 | 22.293 | 0.00741 | 0.01011 | 26.710 |
| 0.60 | 0.03771 | 0.04982 | 24.307 | 0.00807 | 0.01169 | 31.014 |
| 0.70 | 0.03799 | 0.04713 | 19.384 | 0.00763 | 0.01131 | 32.517 |
| 0.80 | 0.04210 | 0.04913 | 14.311 | 0.00611 | 0.00966 | 36.764 |
| 0.90 | 0.04326 | 0.04675 | 7.465 | 0.00343 | 0.00574 | 40.287 |
| 0.99 | 0.04939 | 0.04978 | 0.796 | 0.00039 | 0.00067 | 42.560 |

Based on 10,000 simulated samples

Several other simulations were conducted using various combinations of $\lambda_1$, $\lambda_2$ and $a$ and similar results were found as given in Table 1. Note that since $\lambda_1 = \lambda_2 = 1$, $\mathrm{Corr}(X_1, X_2) = a\lambda_2/\lambda_1 = a$. The empirical MSE was computed for both $\hat{\lambda}_1$, the estimator based on the joint likelihood given in (12), and $\hat{\lambda}_1^*$, the usual maximum likelihood estimator based on the marginal distribution (13). Also computed, were the MLE's of the $\rho$, given as $\hat{\rho} = a\hat{\lambda}_2/\hat{\lambda}_1$ and $\hat{\rho}^* = a\hat{\lambda}_2^*/\hat{\lambda}_1^*$. Percent MSE improvement was computed as $(MSE(\hat{\theta}_1^*) - MSE(\hat{\theta}_1))/MSE(\hat{\theta}_1^*) \cdot 100\%$.

The joint MLE estimator $(\lambda_1, \lambda_2)$ gave MSE improvement over the marginal MLE for all values of $\rho$. Interestingly, percent improvement is a concave function of $\rho$, with maximum occurring at $\rho = 0.5$, giving over 25% improvement. The joint MLE for the correlation coefficient gives monotonically increasing percent improvement over the estimator based on the marginal MLE's, with 44% improvement when $\rho = 0.99$.. Therefore, ignoring the multivariate relationship between $X_1$ and $X_2$ comes at a significant cost with respect to MSE.

## 4. AN INDIRECT LINEAR ASSOCIATION

In the previous section, we directly associated two marginal exponential random variables $X_2$ to $X_1$ through the linear equation $X_2 = aX_1 + Z$. In this section, we indirectly associate the marginal exponentials through the latent variables $Z_1$, $Z_2$ and $X_0$, by letting

$$X_1 = aX_0 + Z_1 \quad \text{and} \quad X_2 = aX_0 + Z_2 \qquad (14),$$

which is another special case of the multivariate gamma given in Definition 1, i.e., $a_1 = a_2 = a$ and $\alpha_1 = \alpha_2 = \alpha_0 = 1$. Notice that from Section 2

$$\text{Cov}(X_1, X_2) = a^2/\lambda_0^2 \qquad (15)$$

and from Section 3

$$P(X_2 = X_1) = P(Z_1 = 0 \text{ and } Z_2 = 0) = P(Z_1 = 0)P(Z_2 = 0) = \frac{a^2\lambda_1\lambda_2}{\lambda_0^2} = \rho > 0. \qquad (16)$$

Note that (16) implies for any sample generated from this population there is a positive probability that the $X_2 = X_1$. Therefore, the correlation $\rho$ can be estimated by computing the sample relative frequency of pairs of data where $x_2 = x_1$. Denote this relative frequency as $\hat{P}(X_2 = X_1)$. We can estimate $\lambda_1$ and $\lambda_2$ using the marginal MLE estimates given in (13) $\hat{\lambda}_1$ and $\hat{\lambda}_2$. Assuming $a$ is known, (15) and (16) give us two different modified method of moments methods to estimate $\lambda_0$ and they are

$$\hat{\lambda}_0^{(1)} = a/\sqrt{m_{12}} \quad \text{and} \quad \hat{\lambda}_0^{(2)} = a\sqrt{\hat{\lambda}_1\hat{\lambda}_2/\hat{P}(X_2 = X_1)}), \qquad (17)$$

where $m_{12}$ is a reasonable sample covariance. If $a$ is unknown then (15) and (16) can be used as a system of two equations and two unknowns the solution of which give the joint method of moments estimates of $a$ and $\lambda_0$. Another set of equations can be derived by computing the sample means $\bar{x}_1^*$ and $\bar{x}_2^*$ computed on only those sample values where $X_1 \neq X_2$. It is easy to see that

$$\begin{aligned} f(x_i|X_1 \neq X_2) &= f_{X_i}(x)/P(X_1 \neq X_2) \\ &= f_{X_i}(x)/(P(Z_1 \neq 0 \; and \; Z_2 \neq 0) \\ &= f_{X_i}(x)(1 - a\lambda_1/\lambda_0)^{-1}(1 - a\lambda_2/\lambda_0)^{-1}, \end{aligned}$$

where $i = 1, 2$. The conditional expected values are equal to

$$E(X_i|X_1 \neq X_2) = [\lambda_i(1 - a\lambda_1/\lambda_0)(1 - a\lambda_2/\lambda_0)]^{-1}.$$

Tables 2 and 3, below, give the Bias and MSE results from simulation studies, conducted in SAS using 10,000 iterations for each correlation value $\rho$. $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the marginal MLEs and $\hat{\rho}$ is found through (16) using the empirical probability estimate. The competing estimators of $\lambda_0$, $\hat{\lambda}_0^{(1)}$ and $\hat{\lambda}_0^{(2)}$, are from (17).

**Table 2: Simulation Study for BVE($\lambda_1 = \lambda_2 = 1, a, n = 50$)**

| | Bias | | | | | |
|---|---|---|---|---|---|---|
| $\rho$ | $\hat{\lambda}_1^*$ | $\hat{\lambda}_2$ | $\hat{\rho}$ | $\hat{\lambda}_0^{(1)}$ | $\hat{\lambda}_0^{(2)}$ | $\% - imp(\hat{\lambda}_0^*)$ |
| 0.01 | 0.01947 | 0.01985 | 0.10051 | -0.95355 | -0.95337 | 0.019 |
| 0.05 | 0.02165 | 0.01920 | 0.04917 | -0.77011 | -0.75495 | 1.968 |
| 0.1 | 0.02157 | 0.01993 | -0.02510 | -0.51827 | -0.44046 | 15.013 |
| 0.2 | 0.01891 | 0.02100 | -0.14689 | -0.06570 | 0.04349 | 33.807 |
| 0.3 | 0.01845 | 0.01914 | -0.20948 | 0.31966 | 0.10320 | 67.717 |
| 0.4 | 0.02046 | 0.02160 | -0.23949 | 0.45078 | 0.05593 | 87.592 |
| 0.5 | 0.02339 | 0.02280 | -0.24889 | 0.35863 | 0.03688 | 89.715 |
| 0.6 | 0.02492 | 0.02286 | -0.23898 | 0.19664 | 0.02955 | 84.970 |
| 0.7 | 0.02118 | 0.01914 | -0.20988 | 0.10093 | 0.02245 | 77.757 |
| 0.8 | 0.02106 | 0.01997 | -0.16045 | 0.07552 | 0.02131 | 71.788 |
| 0.9 | 0.02317 | 0.02377 | -0.08882 | 0.06342 | 0.02256 | 64.421 |
| 0.99 | 0.02201 | 0.02245 | -0.00980 | 0.06073 | 0.02213 | 63.568 |

**Table 3: Simulation Study for BVE($\lambda_1 = \lambda_2 = 1, a,\ n = 50$)**

| | MSE | | | | | |
|---|---|---|---|---|---|---|
| $\rho$ | $\hat{\lambda}_1^*$ | $\hat{\lambda}_2$ | $\hat{\rho}$ | $\hat{\lambda}_0$ | $\hat{\lambda}_0^*$ | $\% - imp(\hat{\lambda}_0^*)$ |
| 0.01 | 0.02158 | 0.02124 | 0.01778 | 0.91258 | 0.91223 | 0.038 |
| 0.05 | 0.02215 | 0.02231 | 0.00982 | 0.65490 | 0.62828 | 4.065 |
| 0.1 | 0.02285 | 0.02206 | 0.00672 | 1.02506 | 0.75806 | 26.047 |
| 0.2 | 0.02241 | 0.02253 | 0.02326 | 1.98727 | 0.89165 | 55.132 |
| 0.3 | 0.02095 | 0.02219 | 0.04553 | 3.13349 | 0.15039 | 95.201 |
| 0.4 | 0.02210 | 0.02263 | 0.06005 | 3.86819 | 0.04370 | 98.870 |
| 0.5 | 0.02254 | 0.02263 | 0.06575 | 6.04188 | 0.02324 | 99.615 |
| 0.6 | 0.02260 | 0.02268 | 0.06182 | 0.74459 | 0.01750 | 97.650 |
| 0.7 | 0.02210 | 0.02197 | 0.04899 | 0.16614 | 0.01578 | 90.500 |
| 0.8 | 0.02215 | 0.02205 | 0.03030 | 0.07274 | 0.01682 | 76.882 |
| 0.9 | 0.02254 | 0.02238 | 0.01099 | 0.05161 | 0.01942 | 62.371 |
| 0.99 | 0.02138 | 0.02143 | 0.00048 | 0.04650 | 0.02105 | 54.720 |

From Tables 2 and 3, we can see that all scale estimators do well both in terms of Bias and MSE. Interestingly, $\hat{\lambda}_0^{(2)}$ gives up to almost 100% MSE improvement and 90% Bias improvement over $\hat{\lambda}_0^{(1)}$.

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